

ON A BINARY SYSTEM OF PRENDIVILLE: THE CUBIC CASE

SHAOMING GUO

ABSTRACT. We prove sharp decoupling inequalities for a class of two dimensional non-degenerate surfaces in \mathbb{R}^5 , introduced by Prendiville [Pre13]. As a consequence, we obtain sharp bounds on the number of integer solutions of the Diophantine systems associated with these surfaces.

1. INTRODUCTION

Let $\Phi(t, s)$ be a homogeneous polynomial of degree three. Consider the two dimensional surface

$$\mathcal{S} = \{(t, s, \Phi_t(t, s), \Phi_s(t, s), \Phi(t, s)) : (t, s) \in [0, 1]^2\}. \quad (1.1)$$

We say that Φ is non-degenerate if it can not be written as $(\mu t + \nu s)^3$ for any $\mu, \nu \in \mathbb{R}$. This is the same as saying that, if we write $\Phi(t, s) = at^3 + bt^2s + cts^2 + ds^3$, then the matrix

$$\begin{pmatrix} 3a & 2b & c \\ b & 2c & 3d \end{pmatrix} \quad (1.2)$$

has rank two. This will be our assumption throughout the present paper.

Consider the following system of Diophantine equations

$$\begin{cases} x_1 + x_2 + \dots + x_r & = x_{r+1} + x_{r+2} + \dots + x_{2r}, \\ y_1 + y_2 + \dots + y_r & = y_{r+1} + y_{r+2} + \dots + y_{2r}, \\ \Phi_t(x_1, y_1) + \dots + \Phi_t(x_r, y_r) & = \Phi_t(x_{r+1}, y_{r+1}) + \dots + \Phi_t(x_{2r}, y_{2r}), \\ \Phi_s(x_1, y_1) + \dots + \Phi_s(x_r, y_r) & = \Phi_s(x_{r+1}, y_{r+1}) + \dots + \Phi_s(x_{2r}, y_{2r}), \\ \Phi(x_1, y_1) + \dots + \Phi(x_r, y_r) & = \Phi(x_{r+1}, y_{r+1}) + \dots + \Phi(x_{2r}, y_{2r}). \end{cases} \quad (1.3)$$

Here r is a positive integer and $x_i, y_i \in \mathbb{N}$ for each $1 \leq i \leq 2r$. For a large integer N , we let $J_r(N)$ denote the number of integer solutions $(x_1, \dots, x_{2r}, y_1, \dots, y_{2r})$ of the system (1.3) with $0 \leq x_i, y_i \leq N$ for each $1 \leq i \leq 2r$. We prove

Theorem 1.1. *For each $r \geq 1$ and each $\epsilon > 0$, we have*

$$J_r(N) \lesssim_{r, \epsilon} N^{2r+\epsilon} + N^{4r-9+\epsilon}. \quad (1.4)$$

Here the implicit constant depends only on r and ϵ . Moreover, up to the arbitrarily small factor ϵ , the exponents of N are sharp.

The lower bounds have been calculated by Parsell, Prendiville and Wooley [PPW13]. Our focus is to obtain the upper bounds (1.4). This will be done via proving a sharp decoupling inequality.

For a measurable set $R \subset [0, 1]^2$ and a measurable function $g : R \rightarrow \mathbb{C}$, define the extension operator associated with \mathcal{S} by

$$E_R g(x) = \int_R g(t, s) e^{itx_1 + isx_2 + i\Phi_t(t, s)x_3 + i\Phi_s(t, s)x_4 + i\Phi(t, s)x_5} dt ds. \quad (1.5)$$

Here $x = (x_1, \dots, x_5)$. For a ball $B = B(c, R) \subset \mathbb{R}^5$ with center c and radius R , we use the weight

$$w_B(x) = (1 + \frac{\|x - c\|}{R})^{-C}, \quad (1.6)$$

where C is a large enough constant whose value will not be specified. For each $2 \leq q \leq p$ and $0 < \delta < 1$, let $B_{p,q}(\delta)$ be the smallest constant such that

$$\|E_{[0,1]^2} g\|_{L^p(w_B)} \leq B_{p,q}(\delta) \left(\sum_{\substack{\Delta: \text{ square in } [0,1]^2 \\ l(\Delta)=\delta}} \|E_{\Delta} g\|_{L^p(w_B)}^q \right)^{1/q}, \quad (1.7)$$

holds for each ball $B \subset \mathbb{R}^5$ of radius δ^{-3} . Via a standard reduction (see for instance Section 2 [BD16-2]), Theorem 1.1 follows from

Theorem 1.2. *We have*

$$B_{9,9}(\delta) \lesssim \left(\frac{1}{\delta}\right)^{2(\frac{1}{2}-\frac{1}{9})+\epsilon}, \quad (1.8)$$

for each $\epsilon > 0$ and $0 < \delta \leq 1$.

The system (1.3) is the cubic case of a system considered by Prendiville [Pre13]. The way that the surface (1.1) and the Diophantine system (1.3) are formulated is slightly different from those in Prendiville [Pre13]. There the surface (1.1) is replaced by

$$\mathcal{S}' = \{(\Phi_{tt}(t, s), \Phi_{ts}(t, s), \Phi_{ss}(t, s), \Phi_t(t, s), \Phi_s(t, s), \Phi(t, s)) : (t, s) \in [0, 1]^2\}. \quad (1.9)$$

That is, the surface \mathcal{S}' is obtained by taking successive partial derivatives of the seed polynomial Φ . However, under the non-degeneracy condition that the matrix (1.2) has rank two, we observe that the vector space $[\Phi_{tt}, \Phi_{ts}, \Phi_{ss}]$ is always the same as $[t, s]$. Hence the system of Diophantine equations associated with the surface \mathcal{S}' is always equivalent to that associated with the surface \mathcal{S} , in the sense that they admit the same number of integer solutions.

To obtain a system analogous to (1.3) of higher degrees, one takes a seed polynomial $\Phi(t, s)$ of degree $k \geq 3$, extracts all the partial derivatives

$$\frac{\partial^{i_1+i_2} \Phi(t, s)}{\partial t^{i_1} \partial s^{i_2}} \quad (i_1 \geq 0, i_2 \geq 0), \quad (1.10)$$

and forms a Diophantine system by using all these partial derivatives. If we take $\Phi(t, s)$ to be the monomial $t^{k_1} s^{k_2}$ with $k_1 \geq k_2 \geq 1$, then we recover the so-called simple binary systems

$$x_1^{i_1} y_1^{i_2} + \dots + x_r^{i_1} y_r^{i_2} = x_{r+1}^{i_1} y_{r+1}^{i_2} + \dots + x_{2r}^{i_1} y_{2r}^{i_2}, \quad \text{with } i_1 \leq k_1, i_2 \leq k_2 \text{ and } (i_1, i_2) \neq (0, 0), \quad (1.11)$$

which appeared in recent work in quantitative arithmetic geometry (Section 4.15 [Tsc09] and [Val11]). Notice that if we take Φ to be a polynomial of degree k that depends only on one variable, then we recover the Vinogradov system

$$x_1^i + \dots + x_r^i = x_{r+1}^i + \dots + x_{2r}^i, \quad \text{with } 1 \leq i \leq k. \quad (1.12)$$

All the systems mentioned above fall into the framework of translation-dilation invariant systems, which are intensively studied in [PPW13]. In our setting, this is reflected in the validity of the parabolic rescaling lemma (Lemma 4.1).

Parsell, Prendiville and Wooley [PPW13] proved (1.4) for $r \geq 21$, using the method of efficient congruencing. In the current paper we prove it for all $r \geq 1$, using the decoupling theory developed in [BD15] and [BDG16]. When intending to generalise our proof to the above binary systems ((1.10) or (1.11)) of degrees higher than three, one encounters enormous difficulties. In comparison, the efficient congruencing method still provides bounds that are almost optimal. We refer to [PPW13] for the precise statement of the corresponding results.

In the end, we mention a few novelties of our proof and explain briefly the potential difficulties that appear when trying to adapt our argument to binary systems of higher degrees.

In decoupling theory, various Brascamp-Lieb inequalities (see (2.2)) play fundamental roles. In order to apply these inequalities, one needs to check a transversality condition (see (2.3)).

When the dimensions and co-dimensions of the surfaces under consideration get higher and higher, checking these transversality conditions will become more and more difficult. In the current paper, we are dealing with a two dimensional surface in \mathbb{R}^5 . To check (2.3), we further develop the idea introduced in [BDG16-2], where a specific two dimensional surface in \mathbb{R}^9 is considered. As currently we are dealing with a class of surfaces, certain algebraic structures need to be explored. For instance, see Subsection 2.2, in particular Lemma 2.3.

A second novelty comes from the iteration argument. Let us compare our approach with those in [BDG16] and [BDG16-2]. The iteration argument in the above two papers relies mainly on the so-called ball-inflation lemmas (see Lemma 3.2). If one considers cubic polynomial surfaces, then usually two such ball-inflation lemmas are required. However, in our case, one such lemma is not available. To overcome this difficulty, we will invoke a square function estimate (see (3.12)). The use of this square function estimate forces us to run the iteration argument slightly differently compared with that in [BDG16] and [BDG16-2]. The main difference is reflected in the use of the quantities that are iterated (see (6.1)).

Notation: Throughout the paper we will write $A \lesssim_v B$ to denote the fact that $A \leq CB$ for a certain implicit constant C that depends on the parameter v . Typically, this parameter is either ϵ or K . The implicit constant will never depend on the scale δ , on the balls we integrate over, or on the function g . It will however most of the times depend on the Lebesgue index p .

We will denote by B_R an arbitrary ball of radius R . We use the following notation for averaged integrals

$$\|F\|_{L^p_\#(w_B)} = \left(\frac{1}{|B|} \int |F|^p w_B\right)^{1/p}.$$

$|A|$ will refer to either the cardinality of A if A is finite, or to its Lebesgue measure if A has positive measure.

Acknowledgements. The author thanks Ciprian Demeter for reading this paper and giving several very useful suggestions.

2. BRASCAMP-LIEB INEQUALITIES

Let m be a positive integer. For $1 \leq j \leq m$, let V_j be a d -dimensional linear subspace of \mathbb{R}^n . Let also $\pi_j : \mathbb{R}^n \rightarrow V_j$ denote the orthogonal projection onto V_j . Define

$$\Lambda(f_1, f_2, \dots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\pi_j(x)) dx, \quad (2.1)$$

for $f_j : V_j \rightarrow \mathbb{C}$. We recall the following theorem due to Bennett, Carbery, Christ and Tao [BCCT10].

Theorem 2.1 ([BCCT10]). *Given $p \geq 1$, the estimate*

$$|\Lambda(f_1, f_2, \dots, f_m)| \lesssim \prod_{j=1}^m \|f_j\|_p \quad (2.2)$$

holds if and only if $np = dm$ and the following Brascamp-Lieb transversality condition is satisfied

$$\dim(V) \leq \frac{1}{p} \sum_{j=1}^m \dim(\pi_j(V)), \text{ for each linear subspace } V \subset \mathbb{R}^n. \quad (2.3)$$

An equivalent formulation of the estimate (2.2) is

$$\left\| \left(\prod_{j=1}^m g_j \circ \pi_j \right)^{1/m} \right\|_q \lesssim \left(\prod_{j=1}^m \|g_j\|_2 \right)^{1/m}, \quad (2.4)$$

with $q = \frac{2n}{d}$. The restriction that $p \geq 1$ becomes $dm \geq n$. The transversality condition (2.3) becomes

$$\dim(V) \leq \frac{n}{dm} \sum_{j=1}^m \dim(\pi_j(V)), \text{ for each subspace } V \subset \mathbb{R}^n. \quad (2.5)$$

Let us be more precise about the parameters in (2.5). We will take $n = 5$ as our surface \mathcal{S} lives in \mathbb{R}^5 . The degree m of multi-linearity will be chosen to be a large number. Our proof will make use of two different values of the parameter d : First of all, we will use $d = 2$, which corresponds to that the surface \mathcal{S} is two-dimensional; secondly, we also need to use $d = 4$, as at certain stage of the proof, we will view \mathcal{S} as a four-dimensional surface in \mathbb{R}^5 . For instance, see Subsection 3.1.

Recall that the surface we are looking at is $(t, s, \Phi_t(t, s), \Phi_s(t, s), \Phi(t, s))$. Its tangent space is spanned by

$$n_1 = (1, 0, \Phi_{tt}, \Phi_{st}, \Phi_t) \text{ and } n_2 = (0, 1, \Phi_{ts}, \Phi_{ss}, \Phi_s). \quad (2.6)$$

Moreover, we denote

$$n_3 = (0, 0, 1, 0, t) \text{ and } n_4 = (0, 0, 0, 1, s). \quad (2.7)$$

We will see from the following Lemma 2.3 that these two vectors span the “second order tangent space”. At a point $\xi \in [0, 1]^2$, let $V_\xi^{(1)}$ be the linear space spanned by $n_1(\xi)$ and $n_2(\xi)$ given in (2.6). Let $V_\xi^{(2)}$ be the linear space spanned by n_1, n_2, n_3, n_4 at the point ξ .

Let $K \in \mathbb{N}$ be a large number. It will be sent to infinity at the end of our proof. A K -square is defined to be a closed square of length $1/K$ inside $[0, 1]^2$. The collection of all dyadic K -squares will be denoted by Col_K .

Proposition 2.2. *Take $\Lambda \in \mathbb{N}$. Denote $m = \Lambda K$. Let R_1, R_2, \dots, R_m be different K -squares from Col_K . For each $1 \leq i \leq m$, choose one point $\xi_i \in R_i$. If we choose Λ sufficiently large, independently on any parameter, then the transversality condition (2.5) with $(d, n) = (2, 5)$ (respectively $(4, 5)$) is satisfied for the collection of spaces $\{V_{\xi_j}^{(1)}\}_{j=1}^m$ (respectively $\{V_{\xi_j}^{(2)}\}_{j=1}^m$).*

We will prove Proposition 2.2 in the following two subsections. How to check the Brascamp-Lieb transversality condition (2.5) seems to have become a big obstacle in obtaining new decoupling inequalities associated with surfaces of high co-dimensions. For instance see [BDG16-2] where a particular two dimensional surface in \mathbb{R}^9 is considered. The forthcoming argument that corresponds to the case $d = 4$ in Subsection 2.2 further develops the idea introduced in [BDG16-2]. From our argument, in particular Lemma 2.3, it will become clear that more algebraic structures need to be understood in order to push our current results to homogeneous polynomials of degrees higher than three. Yet this is not the only obstacle. A further one will be described in the next section.

2.1. Proof of Proposition 2.2: The case $d = 2$. In this subsection we prove the first part of Proposition 2.2. Let $\pi_\xi^{(1)}(V)$ denote the projection of the space V on $V_\xi^{(1)}$. We will show that

$$\dim(V) \leq \frac{5}{2} \dim(\pi_\xi^{(1)}(V)) \text{ almost surely in } \xi. \quad (2.8)$$

Let us assume (2.8) for a moment and see how it implies the transversality condition (2.5). First of all, if we define an exceptional set

$$E_V := \{\xi \in [0, 1]^2 : \dim(V) > \frac{5}{2} \dim(\pi_\xi^{(1)}(V))\}, \quad (2.9)$$

then (2.8) implies that E_V lies inside the zero set of a polynomial of degree less than 10. However, Wongkew’s lemma [Won93] says that the $\frac{10}{K}$ -neighbourhood of the zero set of such a polynomial will intersect at most CK squares in Col_K for some large constant C . The desired

transversality condition (2.5) follows immediately if we choose $\Lambda = 100C$.

Case $\dim(V) = 1$ or 2 . The desired estimate (2.8) is reduced to

$$\dim(\pi_\xi^{(1)}(V)) = 1 \text{ almost surely.} \quad (2.10)$$

Suppose $V = \text{span}\{u\}$ with $u = (u_1, u_2, u_3, u_4, u_5)$. Then (2.10) is equivalent to

$$(u \cdot n_1, u \cdot n_2) \neq (0, 0). \quad (2.11)$$

We argue by contradiction. Suppose $(u \cdot n_1, u \cdot n_2) = (0, 0)$ for every $\xi \in [0, 1]^2$. By checking the constant terms in the polynomials $u \cdot n_1$ and $u \cdot n_2$, we obtain $u_1 = u_2 = 0$. By checking the highest order terms, we obtain $u_5 = 0$. These two facts further imply that the cross product

$$(\Phi_{tt}, \Phi_{st}) \times (\Phi_{ts}, \Phi_{ss}) \quad (2.12)$$

is constantly zero. However, by a direct calculation, this contradicts to the assumption that the polynomial Φ is non-degenerate.

Case $\dim(V) = 3$ or 4 . We need to show that $\dim(\pi_\xi^{(1)}(V)) \geq 2$ almost surely. This is done via a direct calculation. Clearly the case $\dim(V) = 3$ is more difficult. Suppose $V = \text{span}\{u, v, w\}$. Then the dimension of $\pi_\xi^{(1)}(V)$ is equal to the rank of the matrix

$$\begin{pmatrix} u \cdot n_1 & v \cdot n_1 & w \cdot n_1 \\ u \cdot n_2 & v \cdot n_2 & w \cdot n_2 \end{pmatrix} \quad (2.13)$$

We argue by contradiction and suppose that the determinants of all the two by two minors vanish constantly. We look at the two by two minor formed by the first two columns. The determinant of the matrix

$$\begin{pmatrix} u_1 + u_3\Phi_{tt} + u_4\Phi_{st} + u_5\Phi_t & v_1 + v_3\Phi_{tt} + v_4\Phi_{st} + v_5\Phi_t \\ u_2 + u_3\Phi_{ts} + u_4\Phi_{ss} + u_5\Phi_s & v_2 + v_3\Phi_{ts} + v_4\Phi_{ss} + v_5\Phi_s \end{pmatrix} \quad (2.14)$$

vanishes constantly. Denote

$$d_{i,j} := \det \begin{pmatrix} u_i & u_j \\ v_i & v_j \end{pmatrix} \quad (2.15)$$

We first look at the third order term, that is

$$\begin{aligned} & d_{5,4}\Phi_t\Phi_{ss} + d_{5,3}\Phi_t\Phi_{ts} + d_{3,5}\Phi_{tt}\Phi_s + d_{4,5}\Phi_{st}\Phi_s \\ &= \left(d_{3,5}\frac{\partial}{\partial t}\left(\frac{\Phi_t}{\Phi_s}\right) + d_{4,5}\frac{\partial}{\partial s}\left(\frac{\Phi_t}{\Phi_s}\right) \right) \Phi_s^2 \equiv 0. \end{aligned} \quad (2.16)$$

This further implies $d_{3,5} = d_{4,5} = 0$. Moreover, we know $d_{1,2} = 0$ by checking the constant term of the determinant of the matrix (2.14). This further implies that

$$(u_5, v_5, w_5) = (0, 0, 0), \quad (2.17)$$

as otherwise we would derive a contradiction that (u, v, w) , when viewed as a matrix of order 3×5 , has rank two or smaller.

Substitute the identity (2.17) into (2.14), and look at the second order term of the determinant of (2.14). We obtain

$$d_{3,4}\Phi_{tt}\Phi_{ss} - d_{3,4}\Phi_{st}\Phi_{st} \equiv 0. \quad (2.18)$$

By the non-degeneracy assumption on Φ , we obtain that $d_{3,4} = 0$. This, together with (2.17) and $d_{1,2} = 0$, implies that the 3×5 matrix (u, v, w) has rank two or smaller. Contradiction.

2.2. Proof of Proposition 2.2: The case $d = 4$. We let $\pi_\xi^{(2)}(V)$ denote the projection of the space V on $V_\xi^{(2)}$. We need to show that

$$\dim(V) \leq \frac{5}{4} \dim(\pi_\xi^{(2)}(V)) \text{ almost surely.} \quad (2.19)$$

This amounts to calculating the dimension of

$$\{(u \cdot n_1, u \cdot n_2, u \cdot n_3, u \cdot n_4) : u \in V\}. \quad (2.20)$$

Following [BDG16-2], we define linear spaces

$$S_1 = [t, s]; S_2 = [\Phi_t(t, s), \Phi_s(t, s)] \text{ and } S_3 = [\Phi(t, s)]. \quad (2.21)$$

We need the following version of Taylor's formula.

Lemma 2.3. *If $f \in S_3$, then*

$$\Delta f(t, s) \approx f_t(t, s)\Delta t + f_s(t, s)\Delta s + t \cdot f_t(\Delta t, \Delta s) + s \cdot f_s(\Delta t, \Delta s). \quad (2.22)$$

Here $\Delta f(t, s) = f(t + \Delta t, s + \Delta s) - f(t, s)$. The error produced by the approximate identity is a third order homogeneous polynomial in Δt and Δs .

Proof. By linearity, it suffices to consider $f(t, s) = \Phi(t, s)$. We calculate $\Phi(t + \Delta t, s + \Delta s) - \Phi(t, s)$ and view it as a homogeneous polynomial of four variables $t, s, \Delta t$ and Δs . First, we collection the linear terms with respect to Δt and Δs . By the first order Taylor expansion, they are given by $\Phi_t(t, s)\Delta t + \Phi_s(t, s)\Delta s$, which are the former two terms on the right hand side of (2.22). Next, we collect the quadratic terms with respect to Δt and Δs . These terms must be linear in the variables t and s . We apply the first order Taylor expansion again, with the roles of (t, s) and $(\Delta t, \Delta s)$ exchanged, and obtain $t\Phi_t(\Delta t, \Delta s) + s\Phi_s(\Delta t, \Delta s)$, which gives the latter two terms in (2.22). \square

This lemma can be written in the following equivalent way.

$$f(t, s) - f(t_0, s_0) \approx f_t(t_0, s_0)(t - t_0) + f_s(t_0, s_0)(s - s_0) + t_0 \cdot f_t(t - t_0, s - s_0) + s_0 \cdot f_s(t - t_0, s - s_0). \quad (2.23)$$

According to this formula, let us consider

$$f(t, s) = u_1 t + u_2 s + u_3 \Phi_t(t, s) + u_4 \Phi_s(t, s) + u_5 \Phi(t, s). \quad (2.24)$$

At each point $\xi = (t_0, s_0)$, denote $\Delta t = t - t_0$ and $\Delta s = s - s_0$. We define

$$(P_\xi f)(t, s) = f(\xi) + f_t(\xi) \cdot \Delta t + f_s(\xi) \cdot \Delta s + (u_3 + u_5 t_0) \Phi_t(\Delta t, \Delta s) + (u_4 + u_5 s_0) \Phi_s(\Delta t, \Delta s). \quad (2.25)$$

Here we observe that

$$P_\xi f = f \text{ for } f \in S_1 \oplus S_2. \quad (2.26)$$

We further define the canonical projection $\pi_{S_1 \oplus S_2}$ onto the space $S_1 \oplus S_2$. Hence

$$\begin{aligned} (\pi_{S_1 \oplus S_2} P_\xi f)(t, s) &= (f_t(\xi) + (u_3 + u_5 t_0) \Phi_{tt}(-t_0, -s_0) + (u_4 + u_5 s_0) \Phi_{st}(-t_0, -s_0))t \\ &\quad + (f_s(\xi) + (u_3 + u_5 t_0) \Phi_{ts}(-t_0, -s_0) + (u_4 + u_5 s_0) \Phi_{ss}(-t_0, -s_0))s \\ &\quad + (u_3 + u_5 t_0) \Phi_t(t, s) + (u_4 + u_5 s_0) \Phi_s(t, s) \\ &= (u_1 + u_5 \Phi_t(\xi) - u_5 t_0 \Phi_{tt}(\xi) - u_5 s_0 \Phi_{st}(\xi))t + (u_2 + u_5 \Phi_s(\xi) - u_5 t_0 \Phi_{ts}(\xi) - u_5 s_0 \Phi_{ss}(\xi))s \\ &\quad + (u_3 + u_5 t_0) \Phi_t(t, s) + (u_4 + u_5 s_0) \Phi_s(t, s). \end{aligned} \quad (2.27)$$

We can write

$$\begin{aligned} \pi_{S_1 \oplus S_2} P_\xi f &= (u_1 + u_5 \Phi_t(\xi) - u_5 t_0 \Phi_{tt}(\xi) - u_5 s_0 \Phi_{st}(\xi), \\ &\quad u_2 + u_5 \Phi_s(\xi) - u_5 t_0 \Phi_{ts}(\xi) - u_5 s_0 \Phi_{ss}(\xi), u_3 + u_5 t_0, u_4 + u_5 s_0). \end{aligned} \quad (2.28)$$

Recall the choice of the function f in (2.24). Let us compare the vector in (2.28) with the vector in (2.20), which is given by

$$(u_1 + u_3 \Phi_{tt} + u_4 \Phi_{st} + u_5 \Phi_t, u_2 + u_3 \Phi_{ts} + u_4 \Phi_{ss} + u_5 \Phi_s, u_3 + u_5 t_0, u_4 + u_5 s_0). \quad (2.29)$$

By some simple row and column transformations, we see that

$$\begin{aligned} & \dim(\{(u \cdot n_1, u \cdot n_2, u \cdot n_3, u \cdot n_4) : u \in V\}) \\ &= \dim(\{\pi_{S_1 \oplus S_2} P_\xi f : f \in V\}). \end{aligned} \quad (2.30)$$

Hence what we need to show becomes

$$\dim(V) \leq \frac{5}{4} \dim(\{\pi_{S_1 \oplus S_2} P_\xi f : f \in V\}) \text{ almost surely.} \quad (2.31)$$

Case $\dim(V) = 1$. This is the same as the case $\dim(V) = 1$ and $d = 2$.

Case $\dim(V) = 2$. We need to show that $\dim(\pi_{S_1 \oplus S_2} P_\xi(V)) = 2$ almost surely. Argue by contradiction. Suppose $\dim(\pi_{S_1 \oplus S_2} P_\xi(V)) \leq 1$ everywhere. Then

$$V = \pi_{S_1 \oplus S_2}(V) \oplus S_3. \quad (2.32)$$

Let us calculate the projection of S_3 on $S_1 \oplus S_2$. Take

$$f(t, s) = \Phi(t, s) = at^3 + bt^2s + cts^2 + ds^3. \quad (2.33)$$

Hence

$$\pi_{S_1 \oplus S_2} P_\xi f = (-3at_0^2 - 2bt_0s_0 - cs_0^2, -bt_0^2 - 2ct_0s_0 - 3ds_0^2, t_0, s_0) \quad (2.34)$$

As we know that

$$V = \pi_{S_1 \oplus S_2}(V) \oplus S_3, \quad (2.35)$$

if we write $\pi_{S_1 \oplus S_2}(V) = \text{span}\{u\}$ with $u = (u_1, u_2, u_3, u_4)$, then the dimension of $\pi_{S_1 \oplus S_2} P_\xi(V)$ is equal to the rank of the matrix

$$\begin{pmatrix} -3at_0^2 - 2bt_0s_0 - cs_0^2 & -bt_0^2 - 2ct_0s_0 - 3ds_0^2 & t_0 & s_0 \\ u_1 & u_2 & u_3 & u_4 \end{pmatrix} \quad (2.36)$$

For every nonzero vector u , this matrix has rank two almost surely.

Case $\dim(V) = 3$. We need to show that $\dim(\pi_{S_1 \oplus S_2} P_\xi(V)) = 3$ almost surely. Suppose not. Then by taking $\xi = (0, 0)$, we obtain that $\dim(\pi_{S_1 \oplus S_2}(V)) = 2$. Moreover,

$$V = \pi_{S_1 \oplus S_2}(V) \oplus S_3. \quad (2.37)$$

Write $\pi_{S_1 \oplus S_2}(V) = \text{span}\{u, v\}$ with $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$. We need to show that the matrix

$$\begin{pmatrix} -3at_0^2 - 2bt_0s_0 - cs_0^2 & -bt_0^2 - 2ct_0s_0 - 3ds_0^2 & t_0 & s_0 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \end{pmatrix} \quad (2.38)$$

has rank three almost surely. By calculating the determinants of all the 3×3 minors, it is not difficult to see that this is indeed the case.

Case $\dim(V) = 4$. We need to show that $\dim(\pi_{S_1 \oplus S_2} P_\xi(V)) = 4$ almost surely. Similar as above, we prove by contradiction. In the end, we need to show that the matrix

$$\begin{pmatrix} -3at_0^2 - 2bt_0s_0 - cs_0^2 & -bt_0^2 - 2ct_0s_0 - 3ds_0^2 & t_0 & s_0 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} \quad (2.39)$$

has rank four almost surely. Argue by contradiction. Suppose the determinant of the above matrix vanishes constantly. By checking the linear terms in t_0 and s_0 , we obtain that

$$\det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \det \begin{pmatrix} u_1 & u_2 & u_4 \\ v_1 & v_2 & v_4 \\ w_1 & w_2 & w_4 \end{pmatrix} = 0. \quad (2.40)$$

This implies that the two vectors (u_1, v_1, w_1) and (u_2, v_2, w_2) are linearly dependent. That the determinant of the matrix (2.39) vanishes constantly contradicts to the non-degeneracy of the polynomial Φ .

3. CONSEQUENCES OF BRASCAMP-LIEB INEQUALITIES

In this section we state several consequences of the Brascamp-Lieb inequalities obtained in the previous section. They are Lemma 3.2 and Estimate (3.12). These two results will play crucial roles in the iteration argument in Section 6.

A version of Lemma 3.2 is proved in [BDG16], and is a main ingredient in the resolution of the Vinogradov conjecture. As soon as the Brascamp-Lieb condition (2.5) is verified, we may follow the scheme of [BDG16] and conclude Lemma 3.2 from the corresponding Brascamp-Lieb inequalities.

Estimate (3.12) is the main difference between our proof and those of [BDG16] and [BDG16-2]. If we were to follow the scheme of [BDG16] completely, then we would need to prove estimate (3.6) with the Lebesgue index $L_{\#}^{\frac{2p}{5}}$ and balls Δ of radius ρ^{-1} . However, we do not know how to prove such an estimate. This difficulty has already appeared in an earlier work [BDG16-2]. The way out there is to replace the l^6 sum on the left hand side of (3.6) by an l^q sum for certain $q < 6$, that is

$$\frac{1}{|\mathcal{B}|} \sum_{\Delta \in \mathcal{B}} \left[\prod_{i=1}^M \left(\sum_{\substack{R'_i: \text{ square in } R_i \\ l(R'_i)=\rho}} \|E_{R'_i} g\|_{L_{\#}^{\frac{4p}{5}}(w_{\Delta})}^q \right)^{\frac{1}{qM}} \right]^p. \quad (3.1)$$

Here we make a remark on the use of the l^6 sum. Consider the following two dimensional surface in \mathbb{R}^4 given by $(t, s, \Phi_t(t, s), \Phi_s(t, s))$. Denote by \tilde{E} the associated extension operator. It has been proven in [BD16-1] that

$$\|\tilde{E}_{[0,1]^2} g\|_{L^p(w_B)} \leq \left(\frac{1}{\delta}\right)^{2(\frac{1}{2}-\frac{1}{p})+\epsilon} \left(\sum_{\substack{\Delta: \text{ square in } [0,1]^2 \\ l(\Delta)=\delta}} \|\tilde{E}_{\Delta} g\|_{L^p(w_B)}^p \right)^{1/p}, \quad (3.2)$$

for every $2 \leq p \leq 6$, every ball B of radius at least δ^{-2} and every $\epsilon > 0$. Here 6 is exactly the critical exponent for the decoupling inequality (3.2) that we will use in the iteration argument. Changing to the l^q sum as in (3.1) will force us to change the l^p sum in (3.2) to an l^q sum, and prove

$$\|\tilde{E}_{[0,1]^2} g\|_{L^p(w_B)} \leq \left(\frac{1}{\delta}\right)^{2(\frac{1}{2}-\frac{1}{q})+\epsilon} \left(\sum_{\substack{\Delta: \text{ square in } [0,1]^2 \\ l(\Delta)=\delta}} \|\tilde{E}_{\Delta} g\|_{L^p(w_B)}^q \right)^{1/q}, \quad (3.3)$$

for $2 \leq q < p$, which is not true in general. To overcome this difficulty, we prove the “square function” estimate (3.12). In this estimate, instead of an l^2 summation as in usual square functions, we need an l^6 sum out of the above mentioned reasons.

3.1. Ball-inflation: Torsion. In this subsection, we make use of the torsion of the surface \mathcal{S} . In another word, we will view \mathcal{S} as a four dimensional surface in \mathbb{R}^5 . To make clear what it means, we first state a Kakeya inequality.

Lemma 3.1 ([Guth15]). *Let $M = \Lambda K$ where Λ is the same as the one in Proposition 2.2. Let R_1, \dots, R_M be different sets from Col_K . Consider M families \mathcal{P}_j consisting of rectangular boxes P in \mathbb{R}^5 , that we refer to as plates, having the following properties*

1) For each $P \in \mathcal{P}_j$, there exists $\xi_j = (t_j, s_j) \in R_j$ such that four sides of P have lengths equal to $R^{1/2}$ and span $V_{\xi_j}^{(2)}$, while the remaining one side has length R ;

2) all plates are subsets of a ball B_{4R} of radius $4R$.
Then we have the following inequality

$$\oint_{B_{4R}} \left| \prod_{j=1}^M F_j \right|^{\frac{1}{M} \frac{5}{4}} \lesssim_{\epsilon, \nu} R^\epsilon \left[\prod_{j=1}^M \left| \oint_{B_{4R}} F_j \right|^{\frac{1}{M}} \right]^{\frac{5}{4}} \quad (3.4)$$

for each function F_j of the form

$$F_j = \sum_{P \in \mathcal{P}_j} c_P 1_P. \quad (3.5)$$

The implicit constant does not depend on R or c_P .

This Kakeya inequality has the following consequence.

Lemma 3.2. *Let $n = 5$ and $p \geq 8$. Let R_1, \dots, R_M be different squares from Col_K . Let B be an arbitrary ball in \mathbb{R}^n of radius ρ^{-3} . Let \mathcal{B} be a finitely overlapping cover of B with balls Δ of radius ρ^{-2} . For each $g : [0, 1]^2 \rightarrow \mathbb{C}$, we have*

$$\begin{aligned} & \frac{1}{|\mathcal{B}|} \sum_{\Delta \in \mathcal{B}} \left[\prod_{i=1}^M \left(\sum_{\substack{R'_i: \text{ square in } R_i \\ l(R'_i)=\rho}} \|E_{R'_i} g\|_{L_{\#}^{\frac{4p}{5}}(w_\Delta)}^6 \right)^{\frac{1}{6M}} \right]^p \\ & \lesssim_{\epsilon, \nu} \rho^{-\epsilon} \left[\prod_{i=1}^M \left(\sum_{\substack{R'_i: \text{ square in } R_i \\ l(R'_i)=\rho}} \|E_{R'_i} g\|_{L_{\#}^{\frac{4p}{5}}(w_B)}^6 \right)^{\frac{1}{6M}} \right]^p. \end{aligned} \quad (3.6)$$

Proof. We will follow the proof in Bourgain, Demeter and Guth [BDG16], that is, we will derive Lemma 3.2 from Lemma 3.1. In order to apply Lemma 3.1, we need to check that the function $E_{R'_i} g$ is essentially a constant on a plate whose four short sides span the linear space $V_{\xi_j}^{(2)}$ for some $\xi_j \in R'_j$. This follows from Lemma 2.3. We comment here that this is what we meant previously by viewing the surface \mathcal{S} as a four dimensional surface in \mathbb{R}^5 .

In the end, to carry out the proof of Bourgain, Demeter and Guth [BDG16], we just need to observe that $\frac{4p}{5} \geq 6$. This allows us to apply Hölder's inequality

$$\begin{aligned} & \left(\sum_{\substack{R'_i \text{ square in } R_i \\ l(R'_i)=\rho}} \|E_{R'_i} g\|_{L_{\#}^{\frac{4p}{5}}(w_\Delta)}^6 \right)^{\frac{1}{6}} \\ & \lesssim (\#(R_i))^{\frac{1}{6} - \frac{5}{4p}} \left(\sum_{\substack{R'_i \text{ square in } R_i \\ l(R'_i)=\rho}} \|E_{R'_i} g\|_{L_{\#}^{\frac{4p}{5}}(w_\Delta)}^{\frac{4p}{5}} \right)^{\frac{5}{4p}}. \end{aligned} \quad (3.7)$$

Here $\#(R_i)$ denotes the number of squares R'_i inside R_i . For the rest of the details, we refer to [BDG16]. \square

3.2. Ball-inflation: Gaussian curvature. In this subsection, we will make use of the Gaussian curvature of the surface S . By the result of Bennett, Bez, Flock and Lee [BBFL16], we obtain

Theorem 3.3 ([BBFL16]). *Let $M = \Lambda K$. Let $R_1, \dots, R_M \subset [0, 1]^2$ be squares from Col_K . Then*

$$\left\| \left(\prod_{i=1}^M E_{R_i} g_i \right)^{\frac{1}{M}} \right\|_{L^5(B_N)} \lesssim N^\epsilon \left(\prod_{i=1}^M \|g_i\|_{L^2(R_i)} \right)^{\frac{1}{M}}, \quad (3.8)$$

for each $\epsilon > 0$.

From this theorem, and the L^2 orthogonality argument, we obtain

$$\left\| \left(\prod_{i=1}^M E_{R_i} g_i \right)^{\frac{1}{M}} \right\|_{L^5_\#(B_N)} \lesssim N^\epsilon \left(\prod_{i=1}^M \sum_{l(\Delta)=N^{-1/2}} \|E_\Delta g_i\|_{L^2_\#(B_N)}^2 \right)^{\frac{1}{2M}}. \quad (3.9)$$

A randomisation argument further leads to

$$\left\| \left(\prod_{i=1}^M \sum_{l(\Delta)=N^{-1/2}} |E_\Delta g_i|^2 \right)^{\frac{1}{2M}} \right\|_{L^5_\#(B_N)} \lesssim N^\epsilon \left(\prod_{i=1}^M \sum_{l(\Delta)=N^{-1/2}} \|E_\Delta g_i\|_{L^2_\#(B_N)}^2 \right)^{\frac{1}{2M}}. \quad (3.10)$$

We will interpolate this inequality with

$$\left\| \left(\prod_{i=1}^M \sum_{l(\Delta)=N^{-1/2}} |E_\Delta g_i|^q \right)^{\frac{1}{qM}} \right\|_{L^\infty_\#(B_N)} \lesssim N^\epsilon \left(\prod_{i=1}^M \sum_{l(\Delta)=N^{-1/2}} \|E_\Delta g_i\|_{L^\infty_\#(B_N)}^q \right)^{\frac{1}{qM}}, \quad (3.11)$$

for some q that is to be determined. Our goal is to derive

$$\left\| \left(\prod_{i=1}^M \sum_{l(\Delta)=N^{-1/2}} |E_\Delta g_i|^6 \right)^{\frac{1}{6M}} \right\|_{L^p_\#(B_N)} \lesssim N^\epsilon \left(\prod_{i=1}^M \sum_{l(\Delta)=N^{-1/2}} \|E_\Delta g_i\|_{L^{\frac{2p}{5}}_\#(B_N)}^6 \right)^{\frac{1}{6M}}. \quad (3.12)$$

As has been explained above, the l^6 sum on the left hand side of the last expression is crucial for our iteration argument in Section 6. To obtain (3.12) via interpolation, we need to choose $q > 2$. This will make the above interpolation argument different from the one in Bourgain and Demeter [BD15]. We present this interpolation argument in the next subsection.

3.3. How to interpolate decoupling norms. Our goal of this subsection is to prove the estimate (3.12). We adopt the same notation from Bourgain and Demeter [BD15].

The subject in this part is the following decoupling norm. For some small parameter δ , let \mathcal{P}_δ be a finitely overlapping cover of the δ -neighbourhood of our surface \mathcal{S} . For an element $\theta \in \mathcal{P}_\delta$, let f_θ be the frequency restriction of f on θ , that is, $\hat{f}_\theta := \mathbb{1}_\theta \hat{f}$. Define the norm

$$\|f\|_{p,q,\delta} := \left(\sum_{\theta \in \mathcal{P}_\delta} \|f_\theta\|_p^q \right)^{1/q}. \quad (3.13)$$

In [BD15], only $q = 1$ is used. However in our case we have to choose some $q > 1$. This makes the interpolation argument slightly different from that in [BD15].

Let $\phi : \mathbb{R}^5 \rightarrow \mathbb{R}$ be given by

$$\phi(x) = (1 + |x|^2)^{-C}, \text{ for some large enough integer } C. \quad (3.14)$$

Let $N = 1/\delta$. An N -tube T is an $N^{1/2} \times N^{1/2} \times N \times N \times N$ rectangular box in \mathbb{R}^5 , which has dual orientation to some $\theta = \theta(T) \in \mathcal{P}_\delta$. For an N -tube T , let a_T denote the affine transformation that maps T to be unit cube. Define $\phi_T = \phi \circ a_T$.

Definition 3.4 (N -functions and balanced functions). *An N -function is a function $f : \mathbb{R}^5 \rightarrow \mathbb{C}$ such that*

$$f = \sum_{T \in \mathcal{T}(f)} f_T, \quad (3.15)$$

where $\mathcal{T}(f)$ consists of finitely many separated N -tubes T , and moreover

$$|f_T| \lesssim \phi_T \text{ and } \hat{f}_T \subset \theta(T). \quad (3.16)$$

For $\theta \in \mathcal{P}_\delta$, let $\mathcal{T}(f, \theta)$ denote the N -tubes in $\mathcal{T}(f)$ dual to θ . An N -function is called *balanced* if

$$|\mathcal{T}(f, \theta)| \leq 2|\mathcal{T}(f, \theta')| \text{ whenever } \mathcal{T}(f, \theta), \mathcal{T}(f, \theta') \neq \emptyset. \quad (3.17)$$

The reason of introducing balanced functions is that for such a function, the desired estimate (3.12) follows easily from (3.10) and (3.11) via Hölder's inequality.

Definition 3.5 (Counting function). For $q \geq 1$, define

$$\mathcal{T}_q(f) := \left(\sum_{\theta \in \mathcal{P}_\delta} |\mathcal{T}(f, \theta)|^q \right)^{1/q}. \quad (3.18)$$

Lemma 3.6. Fix $q \geq 1$. Every N -function can be written as the sum of $O(\log \mathcal{T}_q(f))$ many balanced N -functions.

This lemma follows from a simple pigeonhole principle. We leave out the details.

Lemma 3.7 (Wave packet decomposition). Assume that f is Fourier supported in \mathcal{N}_δ . Then for each $0 < \lambda \lesssim \|f\|_{\infty, q, \delta}$, there exists an δ^{-1} -function f_λ such that

$$f = \sum_{\lambda \lesssim \|f\|_{\infty, q, \delta}} \lambda f_\lambda, \quad (3.19)$$

and

$$\lambda \delta^{-\frac{8}{p}} \left(\sum_{\theta} |\mathcal{T}(f, \theta)|^{\frac{q}{p}} \right)^{1/q} = \|\lambda f_\lambda\|_{p, q, \theta} \lesssim \|f\|_{p, q, \delta}. \quad (3.20)$$

Proof. Recall that $f = \sum_{\theta \in \mathcal{P}_\delta} f_\theta$. Then we write

$$f_\theta = \sum_{T \in \mathcal{T}_\theta} \langle f_\theta, \varphi_T \rangle \varphi_T, \quad (3.21)$$

where φ_T is an L^2 normalised Schwartz function Fourier localised in θ such that

$$|T|^{1/2} |\varphi_T| \lesssim \phi_T. \quad (3.22)$$

Note that by Hölder's inequality, we have

$$|a_T := \frac{\langle f_\theta, \varphi_T \rangle}{|T|^{1/2}}| \lesssim \|f_\theta\|_\infty \lesssim \|f\|_{\infty, q, \delta}. \quad (3.23)$$

Define

$$f_\lambda = \sum_{\theta} \sum_{T \in \mathcal{T}_\theta: |a_T| \sim \lambda} a_T \lambda^{-1} |T|^{1/2} \varphi_T. \quad (3.24)$$

Now we calculate

$$\|\lambda f_\lambda\|_{p, q, \theta} = \lambda \delta^{-\frac{8}{p}} \left(\sum_{\theta} |\mathcal{T}(f, \theta)|^{\frac{q}{p}} \right)^{1/q} \quad (3.25)$$

Moreover,

$$\|\lambda f_\lambda\|_{p, q, \delta} \lesssim \|f\|_{p, q, \delta}. \quad (3.26)$$

This finishes the proof. \square

Now we are ready to prove (3.12). Denote $f_i = E_{R_i} g_i$. Without loss of generality we can assume that $\|f_i\|_{\frac{2p}{5}, 6, \delta} = 1$ for each $1 \leq i \leq M$. By the wave packet decomposition, we write

$$f_i = \sum_{\lambda \lesssim \|f_i\|_{\infty, 6, \delta}} \lambda f_{i, \lambda}. \quad (3.27)$$

Recall that $\|f_{i, \lambda, \theta}\|_\infty \lesssim 1$ for each θ . This further implies that the left hand side of (3.12) can be controlled by N^C with $f_{i, \lambda}$ in place of f_i . Here C is a large constant that might vary from line to line. Hence the sum over $\lambda \lesssim N^{-C}$ can be well controlled. Next, we observe that $\|f_i\|_{\infty, 6, \delta} \lesssim N^C$. Hence it suffices to consider $\log(\delta^{-1})$ many terms when summing over λ .

For each λ , we decompose $\lambda f_{i, \lambda}$ into balanced functions. According to Lemma 3.6, the number of balanced functions can be controlled by $\log(\delta^{-1})$. This finishes the proof.

4. PARABOLIC RESCALING

In this section we state the following result which is referred to as parabolic rescaling.

Lemma 4.1. *Let $0 < \delta < \sigma \leq 1$. Then for each square $R \subset [0, 1]^2$ with side length σ and each ball $B \subset \mathbb{R}^5$ with radius δ^{-3} we have*

$$\|E_R g\|_{L^p(w_B)} \leq B_{p,q}(\frac{\delta}{\sigma}) \left(\sum_{R' \subset R: l(R')=\delta} \|E_{R'} g\|_{L^p(w_B)}^q \right)^{1/q}. \quad (4.1)$$

The proof of this lemma is standard, see for instance Proposition 7.1 from [BD16-2]. One just needs to observe that our surface \mathcal{S} is translation and dilation invariant, as can be seen via Lemma 2.3.

The parabolic rescaling lemma plays a determinant role in decoupling theory. It is used in every iteration step. First of all, it is used to run the Bourgain-Guth scheme, in order to show the equivalence between the linear and multilinear decoupling inequalities (Theorem 5.1). Secondly, it is used in the iteration scheme in Section 6 to conclude the desired decoupling inequality (1.8).

5. LINEAR VERSUS MULTILINEAR DECOUPLING

In this section we introduce a multi-linear version of the desired decoupling inequality. Recall that K is a large number and $M = \Lambda K$. We denote by $B_{p,q}(\delta, K)$ the smallest constant such that

$$\|(\prod_{i=1}^M E_{R_i} g)^{1/M}\|_{L^p(w_B)} \leq B_{p,q}(\delta, K) \prod_{i=1}^M \left(\sum_{R'_i \subset R_i: l(R'_i)=\delta} \|E_{R'_i} g\|_{L^p(w_B)}^q \right)^{\frac{1}{qM}}. \quad (5.1)$$

holds true for all distinct squares $R_i \in Col_K$, each ball $B \subset \mathbb{R}^9$ of radius δ^{-3} , and each $g : [0, 1]^2 \rightarrow \mathbb{C}$.

By Hölder's inequality, we see that the multi-linear decoupling constant $B_{p,q}(\delta, K)$ can be controlled by the linear decoupling constant $B_{p,q}(\delta)$. It turns out that, in the case $p = q$, the reverse direction also essentially holds true. That is,

Theorem 5.1. *For each $p \geq 2$ and $K \in \mathbb{N}$, there exists $\Omega_{K,p} > 0$ and $\beta(K, p) > 0$ with*

$$\lim_{K \rightarrow \infty} \beta(K, p) = 0, \text{ for each } p, \quad (5.2)$$

such that for each small enough δ , we have

$$B_{p,p}(\delta) \leq \delta^{-\beta(K,p)-2(\frac{1}{2}-\frac{1}{p})} + \Omega_{K,p} \log_K \left(\frac{1}{\delta} \right) \max_{\delta \leq \delta' \leq 1} \left(\frac{\delta'}{\delta} \right)^{2(\frac{1}{2}-\frac{1}{p})} B_{p,p}(\delta', K). \quad (5.3)$$

The proof of this theorem is standard, and is essentially the same as that of Theorem 8.1 from [BD16-2]. Hence we leave it out.

6. ITERATION

In this section, we run the final iteration argument. The consequence of this iteration, combined with Theorem 5.1, will lead to the desired decoupling inequality (1.8).

There will be two terms that are involved in the iteration procedure. They are

$$S_p(q, B^r) := \left\| \left(\prod_{i=1}^M \sum_{J_{i,q} \subset R_i} |E_{J_{i,q}} g_i|^6 \right)^{\frac{1}{6M}} \right\|_{L^p_{\#}(B^r)} \quad (6.1)$$

and

$$D_p(q, B^r) := \left(\prod_{i=1}^M \sum_{J_{i,q} \subset R_i} \|E_{J_{i,q}} g_i\|_{L^p_{\#}(B^r)}^6 \right)^{\frac{1}{6M}} \quad (6.2)$$

Here for a positive number r , we use B^r to denote a ball of radius δ^{-r} . In the notation $J_{i,q}$, the index i indicates that this square lies in R_i , and q indicates that the square $J_{i,q}$ has side length δ^q .

The second term (6.2) appeared both in [BDG16] and in [BDG16-2]. The first term (6.1) is new. As has been pointed out at the beginning of Section 3, we do not know whether the estimate (3.6) still holds true with the Lebesgue space $L_{\#}^{\frac{2p}{5}}$ and balls Δ of radius δ^{-1} . As a substitute, we proved estimate (3.12). As a consequence, this forces us to introduce the term (6.1) to start our iteration argument.

Define $\alpha_1, \alpha_2, \beta_2 \in (0, 1)$ as follows

$$\begin{aligned}\frac{1}{\frac{2p}{5}} &= \frac{\alpha_1}{\frac{4p}{5}} + \frac{1 - \alpha_1}{2}, \\ \frac{1}{\frac{4p}{5}} &= \frac{\alpha_2}{p} + \frac{1 - \alpha_2}{6}, \\ \frac{1}{6} &= \frac{1 - \beta_2}{2} + \frac{\beta_2}{\frac{4p}{5}}.\end{aligned}$$

We will start our iteration with the estimate (3.12)

$$\left\| \left(\prod_{i=1}^M \sum_{J_{i,1} \subset R_i} |E_{J_{i,1}} g_i|^6 \right)^{\frac{1}{6M}} \right\|_{L_{\#}^p(B^2)} \lesssim \delta^{-\epsilon} \left(\prod_{i=1}^M \sum_{J_{i,1} \subset R_i} \|E_{J_{i,1}} g_i\|_{L_{\#}^{\frac{2p}{5}}(B^2)}^6 \right)^{\frac{1}{6M}}. \quad (6.3)$$

This can be summarised as

$$S_p(1, B^2) \lesssim D_{\frac{2p}{5}}(1, B^2). \quad (6.4)$$

By Hölder's inequality, the right hand side of (6.3) can be dominated by

$$D_{\frac{4p}{5}}(1, B^2)^{\alpha_1} D_2(1, B^2)^{1-\alpha_1}. \quad (6.5)$$

By L^2 orthogonality and Hölder's inequality, this can be bounded by

$$\begin{aligned}\delta^{-2(\frac{1}{2}-\frac{1}{6})(1-\alpha_1)} D_{\frac{4p}{5}}(1, B^2)^{\alpha_1} S_2(2, B^2)^{1-\alpha_1} \\ \lesssim \delta^{-2(\frac{1}{2}-\frac{1}{6})(1-\alpha_1)} D_{\frac{4p}{5}}(1, B^2)^{\alpha_1} S_p(2, B^2)^{1-\alpha_1}.\end{aligned} \quad (6.6)$$

We raise the right hand side of (6.6) to the p -th power and sum over all balls B^2 inside a ball B^3 to obtain

$$S_p(1, B^3) \lesssim \delta^{-2(\frac{1}{2}-\frac{1}{6})(1-\alpha_1)} D_{\frac{4p}{5}}(1, B^3)^{\alpha_1} S_p(2, B^3)^{1-\alpha_1}. \quad (6.7)$$

Here we have applied the ball-inflation lemma. The last term $S_p(2, B^3)$ is ready for iteration. We further process the D -term. By Hölder's inequality

$$D_{\frac{4p}{5}}(1, B^3) \lesssim D_6(1, B^3)^{1-\alpha_2} D_p(1, B^3)^{\alpha_2}. \quad (6.8)$$

We further process the former term on the right hand side. By the $l^6 L^6$ decoupling estimate (3.2) for the surface $(t, s, \Phi_t(t, s), \Phi_s(t, s))$ obtained in [BD16-1], we obtain

$$D_6(1, B^3) \lesssim \delta^{-(\frac{1}{2}-\frac{1}{6})-\epsilon} D_6\left(\frac{3}{2}, B^3\right). \quad (6.9)$$

By Hölder's inequality, this can be further bounded by

$$\begin{aligned}D_6(1, B^3) &\lesssim \delta^{-(\frac{1}{2}-\frac{1}{6})-\epsilon} D_6\left(\frac{3}{2}, B^3\right) \\ &\lesssim \delta^{-(\frac{1}{2}-\frac{1}{6})-\epsilon} D_2\left(\frac{3}{2}, B^3\right)^{1-\beta_2} D_{\frac{4p}{5}}\left(\frac{3}{2}, B^3\right)^{\beta_2} \\ &\lesssim \delta^{-(\frac{1}{2}-\frac{1}{6})-\epsilon} \delta^{-3(\frac{1}{2}-\frac{1}{6})} S_p(3, B^3)^{1-\beta_2} D_{\frac{4p}{5}}\left(\frac{3}{2}, B^3\right)^{\beta_2}.\end{aligned} \quad (6.10)$$

In the end, what we have obtained so far can be organised as

$$\begin{aligned} S_p(1, B^3) &\lesssim_{\epsilon} \left(\frac{1}{\delta}\right)^{\epsilon+2(\frac{1}{2}-\frac{1}{6})(1-\alpha_1)+(\frac{1}{2}-\frac{1}{6})\alpha_1(1-\alpha_2)+3(\frac{1}{2}-\frac{1}{6})\alpha_1(1-\alpha_2)(1-\beta_2)} \times \\ &S_p(2, B^3)^{1-\alpha_1} S_p(3, B^3)^{\alpha_1(1-\alpha_2)(1-\beta_2)} D_{\frac{4p}{5}}\left(\frac{3}{2}, B^3\right)^{\alpha_1(1-\alpha_2)\beta_2} D_p(1, B^3)^{\alpha_1\alpha_2}. \end{aligned} \quad (6.11)$$

Now we run this iteration procedure for r many times. For all balls B of radius $\delta^{-2\cdot(\frac{3}{2})^r}$, we have

$$\begin{aligned} S_p(1, B) &\lesssim_{\epsilon, r} \left(\frac{1}{\delta}\right)^{\epsilon+2(\frac{1}{2}-\frac{1}{6})(1-\alpha_1)} \underbrace{\prod_{i=1}^r \left(\frac{1}{\delta}\right)^{2(\frac{3}{2})^i(\frac{1}{2}-\frac{1}{6})\alpha_1(1-\alpha_2)(1-\beta_2)[(1-\alpha_2)\beta_2]^{i-1}}}_{L^2 \text{ orthogonality}} \times \\ &\underbrace{\prod_{i=0}^{r-1} \left(\frac{1}{\delta}\right)^{(\frac{3}{2})^i(\frac{1}{2}-\frac{1}{6})\alpha_1(1-\alpha_2)[(1-\alpha_2)\beta_2]^i}}_{l^6 L^6 \text{ decoupling}} \times S_p(2, B)^{1-\alpha_1} D_{\frac{4p}{5}}\left(\left(\frac{3}{2}\right)^r, B\right)^{\alpha_1[(1-\alpha_2)\beta_2]^r} \\ &\left(\prod_{i=1}^r S_p\left(2\left(\frac{3}{2}\right)^i, B\right)^{\alpha_1(1-\alpha_2)(1-\beta_2)[(1-\alpha_2)\beta_2]^{i-1}}\right) \left(\prod_{i=0}^{r-1} D_p\left(\left(\frac{3}{2}\right)^i, B\right)^{\alpha_1\alpha_2[(1-\alpha_2)\beta_2]^i}\right). \end{aligned} \quad (6.12)$$

Define

$$\begin{aligned} \gamma_0 &= 1 - \alpha_1; \gamma_i = \alpha_1(1 - \alpha_2)(1 - \beta_2)[(1 - \alpha_2)\beta_2]^{i-1}, \text{ for } 1 \leq i \leq r; \\ b_i &= 2 \cdot \left(\frac{3}{2}\right)^i, \text{ for } 0 \leq i \leq r; \\ \tau_r &= \alpha_1[(1 - \alpha_2)\beta_2]^r; \tau_i = \alpha_1\alpha_2[(1 - \alpha_2)\beta_2]^i, \text{ for } 0 \leq i \leq r - 1; \\ w_i &= \frac{1 - \alpha_2}{2\alpha_2}\tau_i, \text{ for } 0 \leq i \leq r - 1. \end{aligned} \quad (6.13)$$

We can write using Hölder

$$D_{\frac{4p}{5}}\left(\left(\frac{3}{2}\right)^r, B\right) \lesssim D_p\left(\left(\frac{3}{2}\right)^r, B\right).$$

With these, the estimate (6.12) becomes

$$\begin{aligned} S_p(1, B) &\lesssim_{r, \epsilon} \left(\prod_{i=0}^r \left(\frac{1}{\delta}\right)^{\epsilon+(\frac{1}{2}-\frac{1}{6})b_i\gamma_i}\right) \left(\prod_{i=0}^{r-1} \left(\frac{1}{\delta}\right)^{(\frac{1}{2}-\frac{1}{6})b_iw_i}\right) \times \\ &\left(\prod_{i=0}^r S_p(b_i, B)^{\gamma_i}\right) \left(\prod_{i=0}^r D_p\left(\frac{b_i}{2}, B\right)^{\tau_i}\right) \end{aligned} \quad (6.14)$$

Recall that in the definition of the quantity D_p we have used an l^6 summation. However as we are eventually aiming at proving an $l^p L^p$ decoupling inequality (for $p = 9$), we also need to introduce the following quantity:

$$\tilde{D}_p(q, B^r) := \left(\prod_{i=1}^M \sum_{R_{i,q} \subset R_i} \|E_{R_{i,q}} g\|_{L_{\#}^p(w_{B^r})}^p\right)^{\frac{1}{pM}}. \quad (6.15)$$

By invoking Hölder's inequality, we get for $p \geq 6$

$$D_p(q, B) \leq \delta^{-2q(\frac{1}{6}-\frac{1}{p})} \tilde{D}_p(q, B). \quad (6.16)$$

Using this and a simple rescaling argument, we can rewrite (6.12) as follows

$$S_p(u, B) \lesssim_{\epsilon, r} \left(\prod_{i=0}^r \left(\frac{1}{\delta} \right)^{\epsilon + u(\frac{1}{2} - \frac{1}{6})b_i \gamma_i} \right) \left(\prod_{i=0}^{r-1} \left(\frac{1}{\delta} \right)^{u(\frac{1}{2} - \frac{1}{6})b_i w_i} \right) \left(\prod_{i=0}^r \left(\frac{1}{\delta} \right)^{u(\frac{1}{6} - \frac{1}{p})b_i \tau_i} \right) \times \\ \left(\prod_{i=0}^r S_p(ub_i, B)^{\gamma_i} \right) \left(\prod_{i=0}^r \tilde{D}_p\left(\frac{ub_i}{2}, B\right)^{\tau_i} \right). \quad (6.17)$$

Here B stands for a ball of radius δ^{-3} , and u is a sufficiently small positive constant such that $u \cdot (\frac{3}{2})^r \leq 1$.

In the end, we iterate (6.17). To iterate, we will dominate each $S_p(ub_i, B)$ again by using (6.17). To enable such an iteration, we need to choose u to be even smaller. Let M be a large integer. Choose u such that

$$[2(\frac{3}{2})^r]^M u \leq 2. \quad (6.18)$$

This allows us to iterate (6.17) M times. When iterating (6.17), we always need to carry the original \tilde{D}_p -terms. To simplify the iteration, we bound all the powers of $\frac{1}{\delta}$ by

$$\left(\sum_{i=0}^{\infty} u(\frac{1}{2} - \frac{1}{6})b_i \gamma_i \right) + \left(\sum_{i=0}^{\infty} u(\frac{1}{2} - \frac{1}{6})b_i w_i \right) + \left(\sum_{i=0}^{\infty} u(\frac{1}{6} - \frac{1}{p})b_i \tau_i \right). \quad (6.19)$$

By a direct calculation,

$$\sum_{j=0}^{\infty} b_j \gamma_j = \frac{-105 + 13p}{2(15 - 10p + p^2)} \quad (6.20)$$

$$\sum_{j=0}^{\infty} b_j w_j = \frac{3(-5 + p)}{2(15 - 10p + p^2)}, \quad (6.21)$$

and

$$\sum_{j=0}^{\infty} b_j \tau_j = \frac{75 - 25p + 2p^2}{15 - 10p + p^2}. \quad (6.22)$$

If we define

$$\lambda_0 := \left(\frac{1}{2} - \frac{1}{6} \right) \left(\frac{-105 + 13p}{2(15 - 10p + p^2)} + \frac{3(-5 + p)}{2(15 - 10p + p^2)} \right) + \left(\frac{1}{6} - \frac{1}{p} \right) \left(\frac{75 - 25p + 2p^2}{15 - 10p + p^2} \right), \quad (6.23)$$

then (6.17) can be rewritten as follows

$$S_p(u, B) \lesssim_{r, \epsilon} \delta^{-\epsilon - u\lambda_0} \left(\prod_{i=0}^r S_p(ub_i, B)^{\gamma_i} \right) \left(\prod_{i=0}^r \tilde{D}_p\left(\frac{ub_i}{2}, B\right)^{\tau_i} \right), \quad (6.24)$$

for every ball B of radius δ^{-3} . Now we have arrived precisely at the estimate (6.51) from [BDG16-2]. The calculation there, from page 27 to page 30, can be repeated line by line, if we replace all the A -terms there by the corresponding S -terms. In the end, we obtain that

$$\log_{\frac{1}{\delta}} B_{9,9}(\delta) \leq \lim_{p \rightarrow 9} \frac{\lambda_0(p)}{\frac{1}{2}(\sum_{j=0}^{\infty} b_j \tau_j(p))}. \quad (6.25)$$

By plugging in the calculation (6.20)–(6.22), we will be able to conclude the desired decoupling inequality (1.8).

REFERENCES

- [BCCT10] Bennett, J., Carbery, A., Christ, M and Tao, T. *Finite bounds for Hölder–Brascamp–Lieb multilinear inequalities*, Math. Res. Lett. 17 (2010), no. 4, 647–666
- [BBFL16] Bennett, J., Bez, N., Flock, T. and Lee, S. *Stability of Brascamp–Lieb constant and applications*, to appear in the Amer. J. Math.
- [BD10] Bostan, A. and Dumas, P. *Wronskians and Linear Independence*, Amer. Math. Monthly 117 (2010), no. 8, 722–727.
- [BD15] Bourgain, J. and Demeter, C. *The proof of the l^2 Decoupling Conjecture*, Annals of Math. 182 (2015), no. 1, 351–389.
- [BD16-1] Bourgain, J. and Demeter, C. *Decouplings for surfaces in R^4* . J. Funct. Anal. 270 (2016), no. 4, 1299–1318.
- [BD16-2] Bourgain, J. and Demeter, C. *Mean value estimates for Weyl sums in two dimensions*, to appear in the J. London Math. Soc.
- [BDG16-2] Bourgain, J., Demeter, C. and Guo, S. *Sharp bounds for the cubic Parsell–Vinogradov system in two dimensions*, arXiv:1608.06346
- [BDG16] Bourgain, J., Demeter, C. and Guth, L. *Proof of the main conjecture in Vinogradov’s mean value theorem for degrees higher than three*, to appear in Annals of Math.
- [BG11] Bourgain, J. and Guth, L. *Bounds on oscillatory integral operators based on multilinear estimates*, Geom. Funct. Anal. 21 (2011), no. 6, 1239–1295
- [Guth15] Guth, L. *A short proof of the multilinear Keakey inequality*. Math. Proc. Cambridge Philos. Soc. 158 (2015), no. 1, 147–153.
- [PPW13] Parsell, S. T., Prendiville, S. M. and Wooley, T. D., *Near-optimal mean value estimates for multidimensional Weyl sums* Geom. Funct. Anal. 23 (2013), no. 6, 1962–2024.
- [Pre13] Prendiville, S. *Solution-free sets for sums of binary forms*. Proc. Lond. Math. Soc. (3) 107 (2013), no. 2, 267–302.
- [Tsc09] Tschinkel, Y. *Algebraic varieties with many rational points*. Arithmetic geometry, 243–334, Clay Math. Proc., 8, Amer. Math. Soc., Providence, RI, 2009.
- [Val11] Van Valckenborgh, K. *Squareful points of bounded heights*. C. R. Math. Acad. Sci. Paris 349 (2011), 603–606.
- [Won93] Wongkew, R. *Volumes of tubular neighbourhoods of real algebraic varieties*, Pacific J. Math. 159 (1993), no. 1, 177–184.

Department of Mathematics, Indiana University, 831 East 3rd St., Bloomington IN 47405
Email address: shaoguo@iu.edu